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A Prelude to Thermal Cosmological Constant in String Thermofield Dynamics

H. Fujisaki,* K. Nakagawa**† and S. Sano*

*Department of Physics, Rikkyo University, Tokyo 171–8501
**Faculty of Pharmaceutical Sciences, Hoshi University, Tokyo 142–8501

Summary

The thermofield dynamics of the dimensionally regularized, one-loop dual symmetric thermal cosmological constant is briefly surveyed after previous publications of ours at any finite temperature in association with the global phase structure of the thermal string ensemble not only for the $D = 26$ closed bosonic thermal string theory but also for the $D = 10$ heterotic thermal string theory.

Building up thermal string theories based upon the thermofield dynamics (TFD) [1] has been recognized as a good practical subject of high energy physics [2–9]. In the present communication, the TFD paradigm of the dimensional regularization of the one-loop dual symmetric thermal cosmological constant is epitomized after previous publications of ourselves [5(1994, 1995, 1997), 7(1994, 1996, 1997), 8] at any finite temperature in full accordance with the thermal stability of modular invariance not only for the $D = 26$ closed bosonic thermal string theory but also for the $D = 10$ heterotic thermal string theory. The global phase structure of the thermal string ensemble is also recapitulated in proper reference to thermal duality [10–13]. The first part provides

†e-mail: nakagawa@hoshi.ac.jp

an abridged overview of the TFD algorithm *à la* ref. [5(1994, 1995)] and ref. [7(1994, 1996, 1997)] in the case of the $D = 26$ closed bosonic thermal string theory. The second part carries out a brief summary of the TFD calculus *à la* ref. [5(1997)] and ref. [8] in the case of the $D = 10$ heterotic thermal string theory.

A. Bosonic string thermofield dynamics

Let us start with the one-loop self-energy amplitude $A(k_1; \zeta_1, \zeta_2; \beta)$ of the massless thermal tensor boson as follows [14]:

$$A(k_1; \zeta_1, \zeta_2; \beta) = -i\kappa^2 \int_{-\infty}^{\infty} d^D p \operatorname{Tr}\left[\Delta^\beta(p) V(k_1; \zeta_1, \bar{\zeta}_1) \Delta^\beta(p) V(k_2; \zeta_2, \bar{\zeta}_2)\right] \tag{1}$$

at any finite temperature in the $D = 26$ closed bosonic thermal string theory based upon the TFD algorithm, where $\kappa, p^\mu, k_r{}^\mu; r = 1, 2$ and $\zeta_r{}^{\mu\nu} = \zeta_r{}^\mu \bar{\zeta}_r{}^\nu; r = 1, 2$ read the coupling constant, loop momentum, external momenta and polarization tensors, respectively, and $V(k; \zeta, \bar{\zeta})$ is referred to as the vertex for the emission or absorption of the massless tensor boson. Here the thermal propagator $\Delta^\beta(p)$ of the free closed bosonic string is written in the form [2]

$$\Delta^\beta(p) = \int_{-\pi}^{\pi} \frac{d\phi}{4\pi} e^{i\phi(L_0 - \bar{L}_0)} \left[\int_0^1 dx\, x^{L_0 + \bar{L}_0 - 2\alpha - 1} \right.$$
$$\left. + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\delta\left[\alpha'/2 \cdot p^2 + 2(n - \alpha)\right]}{e^{\beta|p_0|} - 1} \oint_c d\hat{x}\, \hat{x}^{L_0 + \bar{L}_0 - 2\alpha - 1} \right], \tag{2}$$

where $\overset{[-]}{L}_0 = \alpha'/4 \cdot p^2 + \overset{[-]}{N}$ in which $\overset{[-]}{N}$ reads the number operator of the right-[left-]moving mode, the slope and intercept of the closed string reggeon are $\alpha'/2$ and $2\alpha = (D - 2)/12$, respectively and the contour c is taken as the unit circle around the origin. We are then led to $A(k_1; \zeta_1, \zeta_2; \beta) = A(k_1; \zeta_1, \zeta_2) +$

$\bar{A}(k_1; \zeta_1, \zeta_2; \beta)$ at any value of β, where use has been made of $k_1{}^\mu + k_2{}^\mu = 0$ and $k_1{}^2 = k_2{}^2 = k_1 \cdot \zeta_1 = k_1 \cdot \bar{\zeta}_1 = k_2 \cdot \zeta_2 = k_2 \cdot \bar{\zeta}_2 = 0$. The $D = 26$ zero-temperature amplitude $A(k_1; \zeta_1, \zeta_2)$ is written in the modular invariant fashion as follows [15]:

$$A(k_1; \zeta_1, \zeta_2) = (\pi\kappa)^2(\alpha')^{-D/2}\zeta_1{}^{\mu\nu}\zeta_2{}^{\sigma\rho} \int_F d^2\tau \int_P d^2\nu \tau_2{}^{-D/2}$$
$$\times \exp\left[\pi\tau_2 \cdot \frac{D-2}{6}\right] |f(e^{2\pi i\tau})|^{-2(D-2)}$$
$$\times \left[\left(\frac{\alpha'}{8\pi\tau_2}\right)^2 (\eta_{\mu\nu}\eta_{\sigma\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}) + \left(\frac{\alpha'}{8\pi^2}\right)^2\right.$$
$$\left.\times\ \eta_{\mu\sigma}\eta_{\nu\rho}\left|\frac{\pi}{\tau_2} + \frac{\partial}{\partial\nu}\left\{\frac{\vartheta_1'(\nu-\tau|\tau)}{\vartheta_1(\nu-\tau|\tau)}\right\}\right|^2\right], \qquad (3)$$

where

$$f(w) = \prod_{n=1}^{\infty}(1 - w^n)\ ;\ w = e^{2\pi i\tau}\,, \qquad (4)$$

and $\eta_{\kappa\tau}$ is the space-time metric, ϑ_1 reads the Jacobi theta function, F denotes the fundamental domain of the modular group $SL(2, Z)$ in the complex plane $\tau = \tau_1 + i\tau_2$ and the integration over the complex plane $\nu = \nu_1 + i\nu_2$ is restricted to cover a single parallelogrammatic region P. Since the soft domain $k_1 \simeq 0$ is necessary and sufficient at any finite temperature for the dynamical mass shift of the massless thermal tensor boson, hereafter, the present examination is confined to the asymptotic behaviour of the $D = 26$ temperature-dependent amplitude $\bar{A}(k_1; \zeta_1, \zeta_2; \beta)$ at the low energy limit $k_{10} \simeq 0$. We are then eventually led to the integral representation of the $D = 26$ thermal amplitude $\bar{A}(k_1; \zeta_1, \zeta_2; \beta)$ as follows:

$$\bar{A}(k_1; \zeta_1, \zeta_2; \beta) = 4\pi\left(\frac{\kappa}{4\pi}\right)^2 (\alpha')^{-D/2} \int_0^\infty d\tau_2 \tau_2{}^{-D/2} \exp\left[\pi\tau_2 \cdot \frac{D-2}{6}\right]$$

$$\times \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \int_0^1 \frac{dx_1}{x_1} \theta(x_1 - e^{-2\pi\tau_2}) \sum_{l=1}^{\infty} \exp\left[-\frac{\sigma l^2 \beta^2}{2\tau_2}\right]$$

$$\times \left[f(z_1 z_2)\right]^{-(D-2)} [f(\bar{z}_1 \bar{z}_2)]^{-(D-2)}$$

$$\times \left[\left(\frac{\alpha'}{8\pi\tau_2}\right)^2 \left\{(\zeta_1 \cdot \bar{\zeta}_1)(\zeta_2 \cdot \bar{\zeta}_2) + (\zeta_1 \cdot \bar{\zeta}_2)(\bar{\zeta}_1 \cdot \zeta_2)\right\}\right.$$

$$+ \left(\frac{\alpha'}{2}\right)^2 (\zeta_1 \cdot \zeta_2)(\bar{\zeta}_1 \cdot \bar{\zeta}_2) \left\{\frac{1}{4\pi\tau_2} + \frac{z_2}{(1-z_2)^2} + \Omega'_{12}(z_2, z_1 z_2)\right\}$$

$$\left.\times \left\{\frac{1}{4\pi\tau_2} + \frac{\bar{z}_2}{(1-\bar{z}_2)^2} + \Omega'_{12}(\bar{z}_2, \bar{z}_1 \bar{z}_2)\right\}\right] \tag{5}$$

at $k_{10} \simeq 0$. It is of interest to note that the thermal amplitude $\mathrm{Im}\bar{A}(k_1; \zeta_1, \zeta_2; \beta)$ vanishes identically at $k_{10} \simeq 0$ for any finite temperature .

All we have to do is now reduced to carrying out the regularization of the $D = 26$ thermal amplitude $A(k_1; \zeta_1, \zeta_2; \beta)$ at $k_{10} \simeq 0$. After all, the $D = 26$ one-loop mass shift $A(k_1 \simeq 0; \zeta_1, \zeta_2; \beta)$ of the dilaton, graviton and antisymmetric tensor boson is described at any finitie temperature in the standard fashion as [15]

$$A(k_1 \simeq 0; \zeta_1, \zeta_2; \beta) = (\pi\kappa)^2 (\alpha')^{-D/2} \zeta_1{}^{\mu\nu} \zeta_2{}^{\sigma\rho} \int_F d^2\tau \int_P d^2\nu \tau_2{}^{-D/2}$$

$$\times \exp\left[\pi\tau_2 \cdot \frac{D-2}{6}\right] |f(e^{2\pi i\tau})|^{-2(D-2)} \sum_{m,n \in Z} \exp\left[-\frac{\beta^2}{4\pi\alpha'\tau_2} |m\tau - n|^2\right]$$

$$\times \left\{D_{\mu\nu\sigma\rho}(\nu, \tau) + G_{\mu\nu\sigma\rho}(\nu, \tau) + T_{\mu\nu\sigma\rho}(\nu, \tau)\right\} , \tag{6}$$

where

$$\begin{pmatrix} D_{\mu\nu\sigma\rho}(\nu, \tau) \\ G_{\mu\nu\sigma\rho}(\nu, \tau) \\ T_{\mu\nu\sigma\rho}(\nu, \tau) \end{pmatrix} = \left(\frac{\alpha'}{8\pi\tau_2}\right)^2$$

$$\times \begin{pmatrix} \eta_{\mu\nu}\eta_{\sigma\rho} \\ 1/2 \cdot \left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}\right)\left\{(\tau_2/\pi)^2 \left|\partial_{\nu'}{}^2 G(\nu', \tau)\right|^2 + 1\right\} \\ 1/2 \cdot \left(\eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma}\right)\left\{(\tau_2/\pi)^2 \left|\partial_{\nu'}{}^2 G(\nu', \tau)\right|^2 - 1\right\} \end{pmatrix} \quad (7)$$

and $G(\nu', \tau)$ is often referred to as the bosonic world sheet propagator on the torus in which $D_{\mu\nu\sigma\rho}(\nu, \tau)$, $G_{\mu\nu\sigma\rho}(\nu, \tau)$ and $T_{\mu\nu\sigma\rho}(\nu, \tau)$ depict the relative weights of the dilaton, graviton and antisymmetric tensor boson contribution, respectively. It is almost needless to mention that the term with $m = n = 0$ is identical in every detail to the $D = 26$ zero-temperature amplitude $A(k_1; \zeta_1, \zeta_2)$ and that $A(k_1 \simeq 0; \zeta_1, \zeta_2; \beta)$ is manifestly modular invariant and double periodic and consequently free of ultraviolet divergences at $\tau_2 \sim 0$ and $|\nu| \sim \infty$ for any value of β and D.

The standard integral representation (6) is still annoyed with infrared divergences near the endpoints $\tau_2 \sim \infty$ and $|\nu| \sim 0$, however, unless $D < 2$. The regularization of the ν integration at $|\nu| \sim 0$ is indeed brought to realization in the perspicuously modular invariant fashion [16, 17]. Moreover, the infrared divergence at $\tau_2 \sim \infty$ can be remedied through the dimensional regularization of $\Lambda(\beta)$ *à la* ref. [3] in the sense of analytic continuation which is transparently modular invariant as well as double periodic. The dimensionally regularized, $D = 26$ one-loop dual symmetric mass shift $\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \beta)$ of the dilaton, graviton and antisymmetric tensor boson is then eventually reduced to

$$\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \beta) = -4\pi(\pi\kappa)^2(4\pi^2)^{(D-1)/2}\zeta_1{}^{\mu\nu}\zeta_2{}^{\sigma\rho}\hat{\Lambda}(\beta)$$

$$\times \left\{\tilde{D}_{\mu\nu\sigma\rho} + \tilde{G}_{\mu\nu\sigma\rho} + \tilde{T}_{\mu\nu\sigma\rho}\right\}; \; D = 26\,, \quad (8)$$

where $\tilde{D}_{\mu\nu\sigma\rho} = (\alpha'/8\pi)^2\eta_{\mu\nu}\eta_{\sigma\rho}$, $\tilde{G}_{\mu\nu\sigma\rho} = 0$, $\tilde{T}_{\mu\nu\sigma\rho} = (\alpha'/8\pi)^2(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$ and $\hat{\Lambda}(\beta)$ reads the dimensionally regularized, $D = 26$ one-loop dual symmetric thermal cosmological constant as follows:

$$\hat{\Lambda}(\beta) = -\frac{\sqrt{\pi\alpha'}}{\beta}(4\pi\alpha')^{-D/2}\sum_{m,n\in Z}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp\left[2\pi i m n \tau_1\right]$$
$$\times\left(\frac{\beta^2}{4\pi^2\alpha'}m^2+\frac{4\pi^2\alpha'}{\beta^2}n^2-\frac{D-2}{6}\right)^{(D-1)/2}$$
$$\times\Gamma\left[-\frac{D-1}{2},\pi\sqrt{1-\tau_1{}^2}\left(\frac{\beta^2}{4\pi^2\alpha'}m^2+\frac{4\pi^2\alpha'}{\beta^2}n^2-\frac{D-2}{6}\right)\right]; \tag{9}$$

D=26, which is asymptotically reduced to

$$\hat{\Lambda} = \frac{1}{2}(\pi\alpha')^{-D/2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Gamma\left(-\frac{D}{2}, -4\pi\sqrt{1-\tau_1{}^2}\right);\ D = 26 \tag{10}$$

at zero temperature . It is a matter of course that $\tilde{D}_{\mu\nu\sigma\rho}$, $\tilde{G}_{\mu\nu\sigma\rho}$ and $\tilde{T}_{\mu\nu\sigma\rho}$ describe the factors of the dilaton, graviton and antisymmetric tensor boson contriburion, respectively. Thus, the dimensionally regularized, $D = 26$ one-loop TFD self-energy amplitude $\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \beta)$ satisfies the thermal duality symmetry

$$\beta\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \beta) = \tilde{\beta}\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \tilde{\beta});\ D = 26 \tag{11}$$

for any value of β owing to the thermal duality relation $\beta\hat{\Lambda}(\beta) = \tilde{\beta}\hat{\Lambda}(\tilde{\beta})$; $D = 26$, where $\tilde{\beta} = 4\pi^2\alpha'/\beta$. In accordance, the dimensionally regularized, $D = 26$ thermal amplitude $\hat{A}(k_1 \simeq 0; \zeta_1, \zeta_2; \beta)$ yields the non-vanishing one-loop dual symmetric mass shift for the dilaton and antisymmetric tensor boson which is literally proportional at any finite temperature to the dimensionally regularized, one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$. The dimensionally regularized, one-loop dual symmetric mass shift of the graviton on the contrary is self-consistently guaranteed to vanish identically at any finite temperature. These observations based upon the TFD paradigm are in consonance with the thermal stability of renormalizability, factorizability, duality and gauge invariance which is in turn substantiated at the soft limit $k_1 \simeq 0$ as a direct consequence of the thermal stability of both modular invariance and double periodicity. It is parenthetically argued that the so-called $i\varepsilon$ prescription $D + i\varepsilon$ is adopted under explicit evaluation of the dimensionally regularized, $D = 26$ thermal cosmological constant $\hat{\Lambda}(\beta)$ in full agreement with the off-shell continuation algorithm which inevitably yields finite but complex values for $\hat{\Lambda}(\beta)$ in direct association with the nonvanishing decay rate of the tachyonic thermal vacuum. It will indeed be of practical interest to note that the dimensionally regularized, $D = 26$ zero-temperature cosmological constant (10) develops the imaginary part [3]

$$-\mathrm{Im}\hat{\Lambda} = \frac{1}{2}(\pi\alpha')^{-D/2}\frac{\pi}{\Gamma(D/2+1)};\ D = 26 \qquad (12)$$

as a simple and natural consequence of the $i\varepsilon$ paradigm mentioned above.

Let us next turn our attention to the singularity structure of the dimensionally regularized, $D = 26$ one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$. The position of the singularity $\beta_{|m|,|n|}$ is determined by solving $\beta/\tilde{\beta} \cdot m^2 + \tilde{\beta}/\beta \cdot n^2 - 4 = 0$ for every allowed (m, n) in eq.(9). In particular, $\beta_{1,0}$ and $\tilde{\beta}_{1,0}$ form the leading branch points at $\beta_H = 4\pi\sqrt{\alpha'}$ and $\tilde{\beta}_H = \pi\sqrt{\alpha'}$, respectively, where $\beta_H[\tilde{\beta}_H]$ reads the inverse [dual] Hagedorn temperature. Moreover, there appears the self-dual leading branch point at $\beta_{2,0} = \tilde{\beta}_{2,0} = \beta_0 = 2\pi\sqrt{\alpha'}$ as an inevitable consequence of the thermal duality symmetry. In addition, $\beta_{1,1}$ and $\tilde{\beta}_{1,1}$ yield the non-leading branch points at $(\sqrt{3} + 1)\pi\sqrt{2\alpha'}$ and $(\sqrt{3} - 1)\pi\sqrt{2\alpha'}$, respectively. Finally, all the residual secondary branch points at $\beta_{|m|,0}$ $[\tilde{\beta}_{|m|,0}]$ with $m = \pm 3; \pm 4; \cdots$ are, of course, removed onto the unphysical sheet of the physical $\tilde{\beta}$ $[\beta]$ channel across the leading branch cut mentioned above. Let us recapitulatively touch upon the statistical ensemble of the $D = 26$ closed bosonic thermal string. The thermodynamical properties of the bosonic thermal string excitation can be analyzed through the microcanonical ensemble paradigm outside the analyticity domain of the canonical ensemble. Substantial use is made of the thermal duality relation not only for the canonical region but also for the microcanonical region. There will then exist four phases as follows: I) the β channel canonical phase in the range $4\pi\sqrt{\alpha'} = \beta_H \leqq \beta < \infty$, II) the dual $\tilde{\beta}$ channel canonical phase in the range $0 < \beta \leqq \tilde{\beta}_H = \beta_H/4$, III) the β channel microcanonical domain $\beta_H/2 = \beta_0 \leqq \beta < \beta_H$ and IV) the dual $\tilde{\beta}$ channel microcanonical domain $\tilde{\beta}_H < \beta \leqq \tilde{\beta}_0 = \beta_0$. As a consequence, it will be possible to claim that the so-called maximum temperature of the $D = 26$ closed bosonic thermal string theory is asymptotically described in the sense of the thermal duality relation as the self-dual temperature ${\beta_0}^{-1} = {\tilde{\beta}_0}^{-1} = 2 \cdot {\beta_H}^{-1} = 1/2\pi\sqrt{\alpha'}$.

B. Heterotic string thermofield dynamics

Let us start with the one-loop cosmological constant $\Lambda(\beta)$ as follows [14]:

$$\Lambda(\beta) = \frac{\alpha'}{2} \lim_{\mu^2 \to 0} \mathrm{Tr}\left[\int_{\infty}^{\mu^2} dm^2 \left(\Delta_B^{\beta}(p, P; m^2) + \Delta_F^{\beta}(p, P; m^2)\right)\right] \tag{13}$$

at any finite temperature in the $D = 10$ heterotic thermal string theory based upon the TFD algorithm, where α' means the slope parameter, p^{μ} reads loop momentum and P^I lie on the root lattice $L = \Gamma_8 \times \Gamma_8$ for the exceptional group $G = E_8 \times E_8$. Here the thermal propagator $\Delta^{\beta}_{B[F]}(p, P; m^2)$ of the free closed bosonic [fermionic] string is expressed at $D = 10$ as [2]

$$\Delta^{\beta}_{B[F]}(p, P; m^2) = \int_{-\pi}^{\pi} \frac{d\phi}{4\pi}\, \mathrm{e}^{i\phi\left(N - \alpha - \bar{N} + \bar{\alpha} - 1/2\cdot\sum_{I=1}^{16}(P^I)^2\right)}$$

$$\times \left(\left[\ {}_{[-]}^{+} \int_0^1 dx + \frac{1}{2}\sum_{n=0}^{\infty} \frac{\delta\left[\alpha'/2 \cdot p^2 + \alpha'/2 \cdot m^2 + 2(n - \alpha)\right]}{\mathrm{e}^{\beta|p_0|} \ {}_{[+]}^{-} \ 1} \oint_c dx\right]\right.$$

$$\times\, x^{\alpha'/2\cdot p^2+N-\alpha+\bar{N}-\bar{\alpha}+1/2\cdot\sum_{I=1}^{16}(P^I)^2+\alpha'/2\cdot m^2-1}\Bigg), \tag{14}$$

where N $[\bar{N}]$ denotes the number operator of the right- [left-] mover, the intercept parameter α $[\bar{\alpha}]$ is fixed at $\alpha = 0$ $[\bar{\alpha} = 1]$ and the contour c is taken as the unit circle around the origin. We are then eventually led to the modular parameter integral representation of $\Lambda(\beta)$ at $D = 10$ as follows [10]:

$$\begin{aligned}\Lambda(\beta) =& -8(2\pi\alpha')^{-D/2}\int_E \frac{d^2\tau}{2\pi\tau_2{}^2}(2\pi\tau_2)^{-(D-2)/2}\\ &\times e^{2\pi i\bar{\tau}}\left[1+480\sum_{m=1}^{\infty}\sigma_7(m)\bar{z}^m\right]\\ &\times\prod_{n=1}^{\infty}(1-\bar{z}^n)^{-D-14}\left(\frac{1+z^n}{1-z^n}\right)^{D-2}\sum_{\ell\in Z;\text{odd}}\exp\left[-\frac{\beta^2}{4\pi\alpha'\tau_2}\ell^2\right],\end{aligned} \tag{15}$$

where $\overset{[-]}{\tau} = \tau_1 \underset{[-]}{+} i\tau_2$, $z = xe^{i\phi} = e^{2\pi i\tau}$, $\bar{z} = xe^{-i\phi} = e^{-2\pi i\bar{\tau}}$, E means the half-strip integration region in the τ plane, i.e. $-1/2 \leqq \tau_1 \leqq 1/2$; $\tau_2 > 0$, and full use has been made of an explicit expression of the theta function $\Theta_{\Gamma_8\times\Gamma_8}$ of the root lattice $\Gamma_8 \times \Gamma_8$. Accordingly, the $D = 10$ thermal amplitude $\beta\Lambda(\beta)$ is identical in every detail with the "E-type" representation of the thermo-partition function $\Omega_h(\beta)$ of the heterotic string in ref. [10]. The "E-type" thermal amplitude $\Lambda(\beta)$ is not modular invariant and annoyed with ultraviolet divergences at the endpoint $\tau_2 \sim 0$ for $\beta \leqq \beta_H = (2+\sqrt{2})\pi\sqrt{\alpha'}$, where β_H reads the inverse Hagedorn temperature of the heterotic thermal string. It is to be remembered that the thermal amplitude $\Lambda(\beta)$ is infrared convergent at the upper limit $\tau_2 \to \infty$ for any value of β, which in turn reflects the absence of the tachyonic mode.

Let us now postulate the one-loop dual symmetric thermal cosmological constant $\bar{\Lambda}(\beta;\, D)$ at any space-time dimension D as an integral over the fundamental domain F, i.e. $-1/2 \leqq \tau_1 \leqq 1/2$; $\tau_2 > 0$; $|\tau| > 1$, of the modular group

$SL(2, Z)$ as follows [10]:

$$\bar{\Lambda}(\beta; D) = \frac{2}{\beta}(2\pi\alpha')^{-D/2} \sum_{(\sigma,\rho)} \int_F \frac{d^2\tau}{2\pi\tau_2{}^2} B(\bar{\tau}, \tau; D) A_{\sigma\rho}(\tau; D) D_{\sigma\rho}(\bar{\tau}, \tau; \beta) , \tag{16}$$

where

$$B(\bar{\tau}, \tau; D) = (2\pi\tau_2)^{-(D-2)/2} \bar{z}^{-(D+14)/24} z^{-(D-2)/24}$$
$$\times \left[1 + 480 \sum_{m=1}^{\infty} \sigma_7(m) \bar{z}^m \right] \prod_{n=1}^{\infty} (1 - \bar{z}^n)^{-D-14} (1 - z^n)^{-D+2} , \tag{17}$$

$$\begin{pmatrix} A_{+-}(\tau; D) \\ A_{-+}(\tau; D) \\ A_{--}(\tau; D) \end{pmatrix} = 8 \left(\frac{\pi}{4}\right)^{(D-2)/6} \begin{pmatrix} -\left[\theta_2(0,\tau)/\theta_1'(0,\tau)^{1/3}\right]^{(D-2)/2} \\ -\left[\theta_4(0,\tau)/\theta_1'(0,\tau)^{1/3}\right]^{(D-2)/2} \\ \left[\theta_3(0,\tau)/\theta_1'(0,\tau)^{1/3}\right]^{(D-2)/2} \end{pmatrix} , \tag{18}$$

$$D_{\sigma\rho}(\bar{\tau}, \tau; \beta) = C_\sigma^{(+)}(\bar{\tau}, \tau; \beta) + \rho C_\sigma^{(-)}(\bar{\tau}, \tau; \beta) , \tag{19}$$

$$C_\sigma^{(\gamma)}(\bar{\tau}, \tau; \beta) = (4\pi^2\alpha'\tau_2)^{1/2} \sum_{(p,q)} \exp\left[-\frac{\pi}{2} \left(\frac{\beta}{\tilde{\beta}} p^2 + \frac{\tilde{\beta}}{\beta} q^2 \right) \tau_2 + i\pi p q \tau_1 \right] , \tag{20}$$

$\tilde{\beta} = 2\pi^2\alpha'/\beta$, the signature σ, ρ and γ read $\sigma, \rho = +,-; -,+; -,-$ and $\gamma = +,-$, respectively, the summation over p $[q]$ is restricted by $(-1)^p = \sigma$ $[(-1)^q = \gamma]$ and the explicit use has been made of the Jacobi theta functions $\theta_j(0, \tau)$; $j = 1, 2, 3, 4$ as well as the Poisson resummation formula. It is almost needless to mention that the $D = 10$ dual symmetric thermal amplitude $\beta\bar{\Lambda}(\beta; D = 10)$ is literally reduced to the "D-type" representation of the thermo-partition function $\Omega_h(\beta)$ in ref. [10], which in turn guarantees $\bar{\Lambda}(\beta; D = 10) = \Lambda(\beta)$ as expected from self-consistency. The scalar product $\sum_{(\sigma,\rho)} A_{\sigma\rho} D_{\sigma\rho}$ is invariant under permutations of the signature, irrespective of the values of β and D, not only for the shifting transformation $\tau \to \tau + 1$ but also for the inversion $\tau \to -\tau^{-1}$. In addition , $B(\bar{\tau}, \tau; D)$ is invariant, irrespective of the value of D, under the action of any modular transformation. Accordingly, the dual symmetric thermal amplitude $\bar{\Lambda}(\beta; D)$ is manifestly modular invariant and free of ultraviolet divergences for any value of β and D. Thus we have succeeded in regularizing the ultraviolet behaviour of the thermal amplitude $\Lambda(\beta)$ through transforming the physical information in the ultraviolet region of the half-strip E into the modular invariant integrand of the dual symmetric thermal amplitude $\bar{\Lambda}(\beta; D)$ over the fundamental domain F. If and only if $D = 10$, on the other hand, the scalar product $\sum_{(\sigma,\rho)} A_{\sigma\rho} D_{\sigma\rho}$ is invariant under the thermal duality transformation $\beta \leftrightarrow \tilde{\beta}$ as a simple and natural consequence of the Jacobi identity $\theta_2^4 - \theta_3^4 + \theta_4^4 = 0$ for the theta functions. We are then led to conclude that the thermal duality relation $\beta\bar{\Lambda}(\beta; D) = \tilde{\beta}\bar{\Lambda}(\tilde{\beta}; D)$ is manifestly broken for the dual symmetric thermal amplitude $\bar{\Lambda}(\beta; D \neq 10)$ off the critical dimension.

Let us recall to our remembrance that $\theta_1'^{-1/3} \sim e^{\pi\tau_2/12}$; $\theta_2 \sim 0$; $\theta_3 \sim 1$; $\theta_4 \sim 1$ near $\tau_2 \to \infty$. The infrared behaviour of the dual symmetric thermal cosmological constant $\bar{\Lambda}(\beta; D)$ is then asymptotically described at $\tau_2 \to \infty$ as

$$\begin{aligned}\bar{\Lambda}(\beta; D) = &- 64\sqrt{2}(8\pi^2\alpha')^{-D/2} \sum_{(p,q)} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi p q \tau_1] \\ &\times \sqrt{\frac{\tilde{\beta}}{\beta}} \int_{\sqrt{1-\tau_1^2}}^{\infty} d\tau_2\, \tau_2^{-(D+1)/2} \\ &\times \exp\left[-\frac{\pi}{2}\tau_2 \left(\frac{\beta}{\tilde{\beta}} p^2 + \frac{\tilde{\beta}}{\beta} q^2 - \frac{5}{12}(D-10) - 6\right)\right], \end{aligned} \tag{21}$$

where $p, q = \pm 1; \pm 3; \pm 5; \cdots$. Uniform convergence of the dual symmetric thermal amplitude $\bar{\Lambda}(\beta; D)$ is assured at any value of β if and only if $D < 2/5$. Infrared convergence of the $D = 10$ dual symmetric TFD amplitude $\bar{\Lambda}(\beta; D = 10)$ is then guaranteed if and only if either $(2 + \sqrt{2})\pi\sqrt{\alpha'} = \beta_H < \beta < \infty$

or $0 < \beta < \tilde{\beta}_H = (2-\sqrt{2})\pi\sqrt{\alpha'}$, where $\tilde{\beta}_H$ reads the inverse dual Hagedorn temperature of the heterotic thermal string, *i.e.* there appear inevitably infrared divergences associated with the presence of the tachyonic mode for $\tilde{\beta}_H \leq \beta \leq \beta_H$ in the modular invariant TFD amplitude $\bar{\Lambda}(\beta;\, D=10)$ which are absent in the original TFD amplitude $\Lambda(\beta)$. The present observation will be summarized as follows [13]: Firstly, the presence of the tachyonic mode is crucial to assure modular invariance of the TFD thermal string amplitude. Secondly, the resultant tachyonic divergence bears the dual relationship to the exponential growth of the state density as a function of the mass. All we have to do is reduced to materializing the dimensional regularization of the infrared behaviour of the $D=10$ dual symmetric TFD amplitude $\bar{\Lambda}(\beta;\, D=10)$. Explicit calculation of the τ_2 integral in eq. (21) is readily performed for the case $D < 2/5$ and yields

$$\begin{aligned}\bar{\Lambda}(\beta;\, D) = &-\frac{128}{\sqrt{\pi}}(16\pi\alpha')^{-D/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi pq\tau_1] \\ &\times\sqrt{\frac{\tilde{\beta}}{\beta}}\left(\frac{\beta}{\tilde{\beta}}p^2+\frac{\tilde{\beta}}{\beta}q^2-\frac{5}{12}(D-10)-6\right)^{(D-1)/2} \\ &\times\Gamma\left[-\frac{D-1}{2},\, \frac{\pi}{2}\sqrt{1-\tau_1{}^2}\left(\frac{\beta}{\tilde{\beta}}p^2+\frac{\tilde{\beta}}{\beta}q^2-\frac{5}{12}(D-10)-6\right)\right],\end{aligned} \tag{22}$$

irrespective of the value of β, where Γ is the incomplete gamma function of the second kind. The right-hand side of eq. (22) indeed satisfies the thermal duality symmetry $\beta\bar{\Lambda}(\beta;\, D) = \tilde{\beta}\bar{\Lambda}(\tilde{\beta};\, D)$ and brings forth the correct analytic continuation from $D < 2/5$ to higher values of D, i.e. $D = 10$. We can therefore define the dimensionally regularized, $D=10$ one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$ by

$$\begin{aligned}\hat{\Lambda}(\beta) = &-\frac{2}{\beta}(8\pi\alpha')^{-(D-1)/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi pq\tau_1] \\ &\times\left(\frac{\beta^2}{2\pi^2\alpha'}p^2+\frac{2\pi^2\alpha'}{\beta^2}q^2-6\right)^{(D-1)/2}\end{aligned}$$

$$\times \Gamma\left[-\frac{D-1}{2}, \frac{\pi}{2}\sqrt{1-\tau_1{}^2}\left(\frac{\beta^2}{2\pi^2\alpha'}p^2+\frac{2\pi^2\alpha'}{\beta^2}q^2-6\right)\right]; \quad D=10, \tag{23}$$

which manifestly satisfies the thermal duality relation $\beta\hat{\Lambda}(\beta) = \tilde{\beta}\hat{\Lambda}(\tilde{\beta})$; $D = 10$ in full accordance with the thermal stability of modular invariance. As a matter of fact, the so-called $i\varepsilon$ prescription $D + i\varepsilon$ is adopted under explicit evaluation of the dimensionally regularized, $D = 10$ dual symmetric TFD amplitude $\hat{\Lambda}(\beta)$ in full agreement with the off-shell continuation algorithm [3,5] which inevitably yields finite but complex values for $\hat{\Lambda}(\beta)$ in direct association with the nonvanishing decay rate of the tachyonic thermal vacuum. The thermal duality symmetry mentioned above immediately yields the asymptotic formula as follows:

$$\hat{\Lambda}(\beta\sim 0) \sim \frac{2\pi^2\alpha'}{\beta^2}\hat{\Lambda}(\beta^{-1}\to 0) = \frac{2\pi^2\alpha'}{\beta^2}\Lambda\,; \quad D=10 \tag{24}$$

for the $D = 10$ heterotic thermal string theory, where Λ literally reads the $D = 10$ zero-temperature, one-loop cosmological constant which is in turn guaranteed to vanish automatically as an inevitable consequence of the Jacobi identity $\theta_2^4 - \theta_3^4 + \theta_4^4 = 0$ for the theta functions. Let us call to our remembrance that the vanishing machinery of the $D = 10$ zero-temperature amplitude Λ is self-evident in the present context due to the absence of the term with $\ell = 0$ [$p = q = 0$] on the right-hand side of eq. (15) [eq. (22) or equivalently eq. (23)]. The present observation is paraphrased as follows [13]: The thermal duality symmetry is inherent to the fact that the total number of degrees of freedom vanishes at extremely high temperature $\beta \sim 0$ in the sense of the modular invariant counting.

Let us next examine the singularity structure of the dimensionally regularized, $D = 10$ one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$. The position of the singularity $\beta_{|p|,|q|}$ is determined by solving $\beta/\tilde{\beta}\cdot p^2 + \tilde{\beta}/\beta\,\cdot$

$q^2 - 6 = 0$ for every allowed (p, q) in eq. (23). We then obtain a set of solutions with $|pq| \leqq 3$ as follows: $\beta_{1,1} = \beta_H = (\sqrt{2} + 1)\pi\sqrt{2\alpha'}$; $\tilde{\beta}_{1,1} = \tilde{\beta}_H = (\sqrt{2} - 1)\pi\sqrt{2\alpha'}$ which form the leading branch points of the square root type at β_H and $\tilde{\beta}_H$, respectively. Moreover, β_H^{-1} $[\tilde{\beta}_H^{-1}]$ represents the lowest temperature singularity for the physical β [dual $\tilde{\beta}$] channel. It is of practical importance to note that there exists no self-dual leading branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$, which yields a striking contrast to the previous argument by ourselves [7(1996, 1997)] for the $D = 26$ closed bosonic thermal string theory in the TFD framework. We are now in the position to touch upon the global phase structure of the $D = 10$ heterotic thermal string ensemble. Analysis is performed through the microcanonical ensemble paradigm outside the analyticity domain of the canonical ensemble [10,18–20]. In particular, substantial use is made of the thermal duality symmetry not only for the canonical region but also for the microcanonical region. There will then appear three phases in the sense of the thermal duality symmetry as follows: (i) the β channel canonical phase in the domain $(2 + \sqrt{2})\pi\sqrt{\alpha'} = \beta_H \leqq \beta < \infty$, (ii) the dual $\tilde{\beta}$ channel canonical phase in the domain $0 < \beta \leqq \tilde{\beta}_H = (2 - \sqrt{2})\pi\sqrt{\alpha'}$ and (iii) the self-dual microcanonical phase in the tachyonic domain $\tilde{\beta}_H < \beta < \beta_H$. In sharp contrast to the global phase structure of the $D = 26$ closed bosonic thermal string ensemble [7(1996,1997)], however, there will occur no effective splitting of the microcanonical region because of the absence of the self-dual branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$. As a consequence, it might still remain to be settled whether the so-called maximum temperature of the $D = 10$ heterotic string excitation is asymptotically described in the proper sense of the thermal duality symmetry as the self-dual temperature $\beta_0^{-1} = \tilde{\beta}_0^{-1}$.

Thus we have succeeded in shedding some new light upon the physical significance of the thermal duality symmetry through the TFD algorithm of the dimensional regularization of the one-loop dual symmetric thermal cosmological constant in full agreement with the thermal stability of modular invariance not only for the $D = 26$ closed bosonic thermal string theory but also for the $D = 10$ heterotic thermal string theory. Let us conclude by emphasizing that the TFD paradigm will deserve more than ephemeral consideration in the thermodynamical investigation of the thermal string ensemble in general.

This communication has been organized in association with the thermal cosmological constant as a brief summary of our previous publications as follows:

Fujisaki H., *Dimensional Regularization of the Closed Bosonic Thermal String*, *Europhys. Lett.* **28** (1994) 623; *Infrared Behaviour of the Closed Bosonic Thermal String*, *Nuovo Cim.* A **108** (1995) 1079; *Thermofield Dynamics of the Heterotic String – Physical Aspects of the Thermal Duality –*, *Europhys. Lett.* **39** (1997) 479.

Fujisaki H. and Nakagawa K., *Infrared Self-Energy Amplitude of the Closed Bosonic Thermal String*, *Europhys. Lett.* **28** (1994) 1; *One-Loop Thermal Tachyon Self-Energy*, *Europhys. Lett.* **28** (1994) 471; *Global Phase Structure of the Closed Bosonic Thermal String*, *Europhys. Lett.* **35** (1996) 493; *Thermofield Dynamics of the Closed Bosonic String – Physical Aspects of the Thermal Duality –*, *Soryushiron Kenkyu* (Kyoto) **94** (1997) D34; *Thermofield Dynamics of the Closed Bosonic String – Thermal Cosmological Constant –*, *Cosmological Constant and the Evolution of the Universe*, ed. by Sato K. *et al.* (Universal

Academy Press, Tokyo, 1996) p. 249.

Fujisaki H., Nakagawa K. and Sano S., *Heterotic String Thermofield Dynamics, Nuovo. Cim. A* **110** (1997) 161; *Comments on Heterotic String Thermofield Dynamics, Soryushiron Kenkyu* (Kyoto) **98** (1998) A12.

Fujisaki H., Sano S. and Nakagawa K., *Thermal Cosmological Constant in the Heterotic String Thermofield Dynamics, Particle Cosmology*, ed. by Sato K. *et al.* (Universal Academy Press, Tokyo, 1998) p. 247.

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